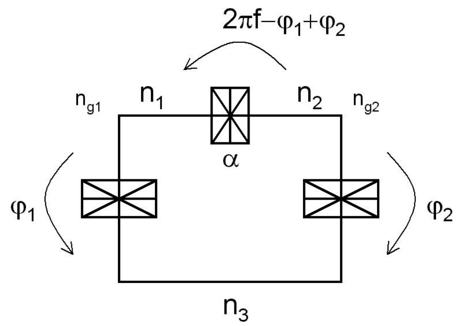
Documentation for Version 1.0



Derivation of the Hamiltonian

Classical equations of motion

$$
\begin{aligned}
I_{\text{left}} &= I_0 \sin(\varphi_1) + C\dot{V} = I_0 \sin(\varphi_1) + C\frac{\phi_0}{2\pi}\ddot{\varphi} && (1)\\
I_{\text{right}} &= I_0 \sin(\varphi_2) + C\frac{\phi_0}{2\pi}\ddot{\varphi}_2 && (2)\\
I_{\text{top}} &= \alpha I_0 \sin(2\pi f - \varphi_1 + \varphi_2) + \alpha C\frac{\phi_0}{2\pi}(\ddot{\varphi}_2 - \ddot{\varphi}_1) && (3)
\end{aligned}
$$

$$I_{\text{left}} = I_{\text{top}} = -I_{\text{right}} \tag{4}$$

Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}(\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2)\\
&= \frac{C}{2}\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_1^2 + \frac{C}{2}\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_2^2 + \alpha\frac{C}{2}\left(\frac{\phi_0}{2\pi}\right)^2 (\dot{\varphi}_1 - \dot{\varphi}_2)^2\\
&\quad +E_J\cos(\varphi_1) + E_J\cos(\varphi_2) + \alpha E_J \cos(\varphi_1 - \varphi_2 - 2\pi f) && (5)
\end{aligned}
$$

The Lagrange equations reproduce the classical eqations.

$$\frac{d}{dt}\left\{\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_i}\right\} - \frac{\partial\mathcal{L}}{\partial\varphi_i} = 0 \tag{6}$$

From the Lagrangian one can derive the canonical variables

$$
\begin{aligned}
q_1 &= \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_1} = C\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_1 + \alpha C\left(\frac{\phi_0}{2\pi}\right)^2 (\dot{\varphi}_1 - \dot{\varphi}_2) && (7)\\
q_2 &= \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_2} = C\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_2 - \alpha C\left(\frac{\phi_0}{2\pi}\right)^2 (\dot{\varphi}_1 - \dot{\varphi}_2) && (8)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\varphi}_1 &= \frac{1}{C\left(\frac{\phi_0}{2\pi}\right)^2}\frac{1+\alpha}{1+2\alpha}q_1 + \frac{1}{C\left(\frac{\phi_0}{2\pi}\right)^2}\frac{\alpha}{1+2\alpha}q_2 && (9)\\
\dot{\varphi}_2 &= \frac{1}{C\left(\frac{\phi_0}{2\pi}\right)^2}\frac{\alpha}{1+2\alpha}q_1 + \frac{1}{C\left(\frac{\phi_0}{2\pi}\right)^2}\frac{1+\alpha}{1+2\alpha}q_2 && (10)
\end{aligned}
$$

With a Legendre transformation we get the Hamiltonian

$$\mathcal{H}(q_1, q_2, \varphi_1, \varphi_2) = \dot{\varphi}_1 q_1 + \dot{\varphi}_2 q_2 - \mathcal{L} \tag{11}$$

$$\begin{aligned}\mathcal{H} &= 4\frac{E_J}{\left(2e\frac{\phi_0}{2\pi}\right)^2}\left(\frac{1+\alpha}{1+2\alpha}q_1^2+\frac{2\alpha}{1+2\alpha}q_1q_2+\frac{1+\alpha}{1+2\alpha}q_2^2\right)\\ &\quad -E_J\cos(\varphi_1)-E_J\cos(\varphi_2)-\alpha E_J\cos(\varphi_1-\varphi_2-2\pi f) \qquad (12)\end{aligned}$$

From the canonical equations

$$\dot{\varphi}_i = \frac{\partial\mathcal{H}}{\partial q_i} \qquad (13)$$

$$\dot{q}_i = -\frac{\partial\mathcal{H}}{\partial\varphi_i} \qquad (14)$$

one gets back the classical equations of motion (4).

Qubit Hamiltonian in the Charge Base

using

$$q_1 = 2e\frac{\phi_0}{2\pi}n_1 \qquad (15)$$

$$q_2 = 2e\frac{\phi_0}{2\pi}n_2 \qquad (16)$$

$$\begin{aligned}\mathcal{H} &= 4E_c\frac{1+\alpha}{1+2\alpha}n_1^2+4E_c\frac{2\alpha}{1+2\alpha}n_1n_2+4E_c\frac{1+\alpha}{1+2\alpha}n_2^2\\ &\quad -E_J\cos(\varphi_1)-E_J\cos(\varphi_2)-\alpha E_J\cos(\varphi_1-\varphi_2-2\pi f) \qquad (17)\end{aligned}$$

with

$$E_J = \frac{I_c\phi_0}{2\pi} \qquad (18)$$

$$E_c = \frac{e^2}{2C} \qquad (19)$$

$$\cos(\varphi_j) = \frac{1}{2}(e^{i\varphi_j}+e^{-i\varphi_j}) \qquad (20)$$

$$e^{i\varphi_j}\left|n_j\right\rangle = \left|n_j+1\right\rangle \qquad (21)$$

$$e^{-i\varphi_j}\left|n_j\right\rangle = \left|n_j-1\right\rangle \qquad (22)$$

$$e^{i(\varphi_j+k)}\left|n_j\right\rangle = e^{ik}\left|n_j+1\right\rangle \qquad (23)$$

$$e^{i(\varphi_j-k)}\left|n_j\right\rangle = e^{-ik}\left|n_j-1\right\rangle \qquad (24)$$

$$\left\langle m_1,m_2\right| - E_J\cos(\varphi_1)\left|n_1,n_2\right\rangle = \delta_{m_2,n_2}\frac{-E_J}{2}(\delta_{m_1,n_1-1}+\delta_{m_1,n_1+1}) \qquad (25)$$

$$\left\langle m_1,m_2\right| - E_J\cos(\varphi_2)\left|n_1,n_2\right\rangle = \delta_{m_1,n_1}\frac{-E_J}{2}(\delta_{m_2,n_2-1}+\delta_{m_2,n_2+1}) \qquad (26)$$

$$\begin{aligned}&\left\langle m_1,m_2\right| - \alpha E_J\cos(\varphi_1-\varphi_2-2\pi f)\left|n_1,n_2\right\rangle =\\ &\left(\delta_{m_1,n_1+1}\delta_{m_2,n_2-1}e^{-i2\pi f}+\delta_{m_1,n_1-1}\delta_{m_2,n_2+1}e^{i2\pi f}\right)\frac{-\alpha E_J}{2} \qquad (27)\end{aligned}$$

Labeling of the charge states:

$$|Q\rangle = |n_2 + (n_1 - 1)N + \frac{1}{2}(N+1)^2\rangle = |n_1, n_2\rangle \tag{28}$$

with

$$-n_{\max} \le n_1, n_2 \le n_{\max} \qquad N = 2n_{\max} + 1 \tag{29}$$

example for $N = 3$, i.e. $n_{\max} = 1$

$$\begin{aligned}
|1\rangle &= |-1,-1\rangle \\
|2\rangle &= |-1,0\rangle \\
|3\rangle &= |-1,1\rangle \\
|4\rangle &= |0,-1\rangle \\
|5\rangle &= |0,0\rangle \\
|6\rangle &= |0,1\rangle \\
|7\rangle &= |1,-1\rangle \\
|8\rangle &= |1,0\rangle \\
|9\rangle &= |1,1\rangle
\end{aligned}$$